First semestral examination 2008
Commutative algebra
M.Math.IInd year
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Answer 6 questions completely. Be brief !

Q 1.

(i) Let $\theta : M \to N$ be an A-module homomorphism. If N is finitely generated and $\theta_P : M_P \to N_P$ is onto, for a fixed prime ideal P, show that there exists $a \notin P$ such that the induced map : $M_a \to N_a$ is onto.

(ii) Give an example to show that even if M, N are finitely generated, then $S^{-1}Hom_A(M,N) \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$ may not be onto.

Q 2.

If M is a finitely generated module over a Noetherian ring A, show $\sqrt{ann(M)} = \bigcap_{P \in Ass(M)} P$.

Q 3.

Consider the ring $A = \mathbf{Z}[2X, X^2, X^3]$. Prove that the ideal P generated by $2X$ and X^2 over A is prime but that P^2 is not P-primary.

Q 4.

(i) Compute $Tor_n^{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ for a prime p for $n = 0$ and $n = 1$.

(ii) Over the ring A of continuous functions from $\mathbf{R}$ to itself with period π, show that the module P of all continuous functions f satisfying $f(x+\pi) = -f(x) \forall x$, is projective but not free.

Q 5.

Let A be a subring of a field K and let $\theta : A \to \Omega$ be a nontrivial ring homomorphism to an algebraically closed field. If $t \in K^*$, show that θ extends to a ring homomorphism either from $A[t]$ or from $A[t^{-1}]$ to Ω.

Q 6.

Let G be a finite group of automorphisms of a ring A.

(i) Prove that A is an integral extension of $A^G := \{a \in A : g(a) = a\}$.

(ii) Show that G acts transitively on the set of prime ideals of A lying over any prime of A^G.

Q 7.

Let A be a Noetherian, local, normal domain in which each non-zero prime ideal is maximal. Show that the maximal ideal of A must be principal.

Q 8.

Show that an Artinian ring has only finitely many maximal ideals.

Q 9.

Let I be a fractional ideal of a domain A. If I is projective as an A-module, prove that I is invertible.

Q 10.

Let $(A, \mathbf{m})$ be a Noetherian, local ring. Using the dimension theorem or otherwise, prove that $\dim A \leq \dim_{A/\mathbf{m}} \mathbf{m}/\mathbf{m}^2$.